JOURNAL OF APPROXIMATION THEORY 27, 146–152 (1979)

An Existence Theorem for Chebyshev Approximation by Interpolating Rationals

DARRELL SCHMIDT

*Department of Mathematics, Marshall University, Huntington, West Virginia 25701**

Communicated by Richard S. Varga

Received March 30, 1978

1. INTRODUCTION

Let $I = [a, b]$ be a compact interval, $C(I)$ be the set of continuous, real-valued functions defined on I, and $p > 0$. Suppose $f(x) = g(x)\, B(x)$ where $g, B \in C(I)$, $g(x) > 0$ on I, and B has finitely many zeros $x_1, x_2, ..., x_s$ in I. Let $R^+(n, m)$ denote the set of all rational functions $R = P/Q$ where P is a polynomial of degree n or less, Q is a polynomial of degree m or less, $P(x) \geqslant 0$ on I, and $Q(x) > 0$ on I. We shall consider the problem of approximating f by elements of the set

$$V(p, n, m) = \{R^pB: R \in R^+(n, m)\}.$$

In particular, an element $(R^*)^p\, B$ of $V(p, n, m)$ is called a best approximation to f from $V(p, n, m)$ if

$$\| f - (R^*)^p\, B \| = \inf_{R \in R^+(n,m)} \| f - R^pB \| \tag{1.1}$$

where $\| \cdot \|$ is the uniform norm over I.

The problem of Chebyshev approximation by interpolating rationals (1.1) (so called because of the inclusion of the factor B) was first considered by J. Williams [5] in the case $n = 0$. That is, his approximants involved reciprocals of polynomials. The question of existence of best approximations proved to be a difficulty in Williams' paper. A later paper by G. D. Taylor and J. Williams [4] gave examples for which best approximations do not exist and established conditions on B and on g which insure the existence of best approximations. The purpose of this paper is to extend Taylor and

* Present address: Department of Mathematical Sciences, Oakland University, Rochester, Mich. 48063.

0021-9045/79/100146–07$02.00/0

Williams' Theorem 3.1 (conditions on B) to the more general setting of approximation from $V(p, n, m)$ with $n \geqslant 0$. We do this by using quite different methods. In addition, we show that the conditions of this theorem are essential when $n \geqslant 2$ and $m \geqslant 2$.

As C. B. Dunham [3] notes, the approximation problem (1.1) can be regarded as a restricted range approximation problem of g by rational functions with respect to a vanishing weight function $| B |$. Dunham [2, 3] has given characterization and uniqueness theorems for similar problems. In Section 3, we state appropriate characterization and uniqueness results for the problem (1.1).

2. Existence of Best Approximations

In this section, we place conditions on B which insure the existence of best approximations to f from $V(p, n, m)$. Essentially the conditions are that the interior zeros of B are of multiplicity less than $2p$ and the endpoint zeros of B are of multiplicity less than p. This will be followed by showing that these conditions are essential when $n \geqslant 2$ and $m \geqslant 2$ in the sense that if B fails to satisfy these conditions, then there is a $g \in C(I)$ with $g(x) > 0$ on I such that $f = gB$ does not have a best approximation from $V(p, n, m)$.

THEOREM 1. *Suppose that for* $\nu = 1,..., s$,

(i) $\overline{\lim}_{x \to x_\nu \pm} | B(x)|/| x - x_\nu |^p = \infty$ *if* $x_\nu = a$ *or* b, *and*

(ii) $\overline{\lim}_{x \to x_\nu} | B(x)|/| x - x_\nu |^{2p} = \infty$ *if* $x_\nu \in (a, b)$.

Then f has a best approximation from $V(p, n, m)$.

Proof. Let

$$d = \inf_{R \in R^+(n,m)} \| f - R^p B \|$$

and select a sequence $\{R_k\}_{k=1}^\infty$ in $R^+(n, m)$ such that $\| f - R_k{}^p B \| \leqslant d + 1$ for all k and $\| f - R_k{}^p B \| \to d$ as $k \to \infty$. We may write $R_k = P_k/Q_k$ where $P_k \in \pi_n$, $Q_k \in \pi_m$, $P_k(x) \geqslant 0$ and $Q_k(x) > 0$ on I, and $\| Q_k \| = 1$. Here π_l denotes the set of all polynomials of degree l or less. Letting $M = \| f \| + d + 1$, we see that

$$| P_k(x)^p B(x)| \leqslant MQ(x)^p \tag{2.1}$$

for all k and all $x \in I$. By (2.1), the P_k are bounded independently of k over any set of $n + 1$ points in $I \backslash \{x_1, ..., x_s\}$. Since π_n is an $(n + 1)$-dimensional Haar subspace of $C(I)$, the P_k are uniformly bounded over I. Thus we may extract convergent subsequences and relabel so that $P_k \to P \in \pi_n$ and $Q_k \to Q \in \pi_m$ uniformly on I as $k \to \infty$. Note that $P(x) \geqslant 0$ and $Q(x) \geqslant 0$ on I, $\| Q \| = 1$, and

$$| P(x)^p B(x)| \leqslant MQ(x)^p \tag{2.2}$$

for all $x \in I$.

Since $\| Q \| = 1$, $Q \not\equiv 0$ and Q can have at most finitely many zeros in I. It remains to show that P/Q is in $R^+(n, m)$ or can be reduced to an element of $R^+(n, m)$. To do this, we need only show that every zero of Q in I is also a zero of P with equal or greater multiplicity. Suppose that x^* is a zero of Q in I. If $x^* \in I\backslash\{x_1, \ldots, x_s\}$, (2.2) implies that

$$0 \leqslant P(x) \leqslant M_1 Q(x)$$

for some $M_1 > 0$ and all x in some neighborhood of x^*. Thus x^* is a zero of P with multiplicity greater than or equal to its multiplicity as a zero of Q. Suppose $x^* = x_\nu \in (a, b)$. Since $Q(x) \geqslant 0$ on I, x_ν is a zero of Q of even multiplicity, say 2μ. By (2.2)

$$\overline{\lim_{x \to x_\nu}} \left| \frac{P(x)}{(x - x_\nu)^{2\mu - 2}} \right|^p \frac{| B(x)|}{| x - x_\nu |^{2p}} \leqslant M \overline{\lim_{x \to x_\nu}} \left| \frac{Q(x)}{(x - x_\nu)^{2\mu}} \right|^p < \infty.$$

Since $\overline{\lim}_{x \to x_\nu} | B(x)|/| x - x_\nu |^{2p} = \infty$, $P(x^l)/(x^l - x_\nu)^{2\mu-2} \to 0$ as $l \to \infty$ for some sequence $\{x^l\}$ which converges to x_ν. As a result, $\lim_{x \to x_\nu} P(x)/(x - x_\nu)^{2\mu-2} = 0$, and x_ν is a zero of P of multiplicity at least $2\mu - 1$. Since $P(x) \geqslant 0$ on I, x_ν must be a zero of P of even multiplicity. Thus x_ν is a zero of P of multiplicity at least 2μ. The case in which $x^* = x_\nu = a$ or b is handled similarly to the case $x^* = x_\nu \in (a, b)$ and is omitted.

Thus there exist $P^* \in \pi_n$ and $Q^* \in \pi_m$ with $P^*(x) \geqslant 0$ and $Q^*(x) > 0$ on I such that

$$\frac{P(x)}{Q(x)} = \frac{P^*(x)}{Q^*(x)}$$

for all $x \in I$ with $Q(x) \neq 0$. Thus $R^* = P^*/Q^* \in R^+(n, m)$. If $x \in I$ and $Q(x) \neq 0$, then

$$| f(x) - R^*(x)^p B(x)| = \lim_{k \to \infty} \left| f(x) - \left[\frac{P_k(x)}{Q_k(x)}\right]^p B(x)\right| \leqslant d.$$

By the continuity of $f - (R^*)^p B$, $\| f - (R^*)^p B \| \leqslant d$, and $(R^*)^p B$ is a best approximation to f from $V(p, n, m)$. Thus the proof of Theorem 1 is complete.

We remark that if $m = 0$, then $V(p, n, m)$ is a closed subset of a finite dimensional subspace of $C(I)$, and thus f has a best approximation from $V(p, n, m)$. So conditions (i) and (ii) can be deleted if $m = 0$. If $m = 1$, then in the proof of Theorem 1, the linear polynomial Q could not vanish in (a, b). Thus if $m = 1$, condition (ii) can be dropped. The next theorem indicates that conditions (i) and (ii) are essential when $n \geqslant 2$ and $m \geqslant 2$.

THEOREM 2. *Let $n \geqslant 2$ and $m \geqslant 2$. If $B(x)$ does not satisfy condition* (i) *of Theorem* 1 *at some $x_\nu = a$ or b or if $B(x)$ does not satisfy condition* (ii) *of Theorem* 1 *at some $x_\nu \in (a, b)$, then there is a $g \in C(I)$ with $g(x) > 0$ on I such that $f = gB$ does not have a best approximation from $V(p, n, m)$.*

Proof. Suppose that for some $x_\eta \in (a, b)$

$$\varlimsup_{x \to x_\eta} | B(x)|/| x - x_\eta |^{2p} < \infty. \tag{2.3}$$

The proof in the case that condition (i) is violated at $x_\eta = a$ or b is similar to the present case and is omitted. In what follows, we shall interpret $(x - x_\eta)^{2p}$ as $[(x - x_\eta)^2]^p$. By (2.3) we may write

$$B(x) = H(x)(x - x_\eta)^{2p}$$

where H is continuous on $[a, x_\eta) \cup (x_\eta, b]$ and $| H |$ is upper semicontinuous at x_η. In addition, we may assume that $| H(x_\eta)| > 0$. For $\epsilon \geqslant 0$, let

$$R_\epsilon(x) = \frac{K(x - x_\eta)^2 + 1}{(x - x_\eta)^2 + \epsilon}$$

where $K > 0$ is sufficiently large that

$$\sup_{\substack{x \in I \\ x \neq x_\eta}} | B(x) R_0(x)^p | = \max_{x \in I} | H(x)| [K(x - x_\eta)^2 + 1]^p > 3 | H(x_\eta)|.$$

Since $B(x_\nu) = 0$, $\nu = 1,..., s$, there is an open interval (α, β) contained in I which is disjoint from $\{x_1,..., x_s\}$ such that $| B(x) R_0(x)^p | > 2 | H(x_\eta)|$ for $x \in (\alpha, \beta)$. Let $l = n + m + 2$ and select l points $\xi_1 < \xi_2 < \cdots < \xi_l$ in (α, β). Then $| B(\xi_i) R_0(\xi_i)^p | > 2 | H(x_\eta)|$, $i = 1,..., l$, and the $B(\xi_i)$ have the same sign. Now let d be such that $| H(x_\eta)| < d < 2 | H(x_\eta)|$.

We now construct the function g. For $i = 1,..., l$, let $g(\xi_i)$ be given by

$$g(\xi_i) B(\xi_i) = R_0(\xi_i)^p B(\xi_i) + (-1)^i d.$$

Since $| R_0(\xi_i)^p B(\xi_i)| > d$, $g(\xi_i) > 0$, $i = 1,..., l$. By the upper semicontinuity of H at x_η, there is a $\delta > 0$ such that $[x_\eta - \delta, x_\eta + \delta] \subseteq I$ and

$$| B(x) R_0(x)^p | = | H(x)| [K(x - x_\eta)^2 + 1]^p \leqslant d$$

for $0 < | x - x_\eta | \leqslant d$. We fix $\epsilon_0 > 0$ and define

$$g(x) = R_{\epsilon_0}(x)^p$$

for $| x - x_\eta | \leqslant \delta$. Then $g(x) > 0$ for $| x - x_\eta | \leqslant \delta$. For $0 < | x - x_\eta | \leqslant \delta$ and any $0 < \epsilon < \epsilon_0$,

$$| g(x) B(x) - R_\epsilon(x)^p B(x)| = | B(x)| \, | R_{\epsilon_0}(x)^p - R_\epsilon(x)^p |$$
$$\leqslant | B(x) R_\epsilon(x)^p | \leqslant | B(x) R_0(x)^p | \leqslant d. \quad (2.4)$$

We finally extend g continuously to all of I so that

$$g(x) > 0 \quad (2.5)$$

for $x \in I$ and

$$| g(x) B(x) - R_0(x)^p B(x)| \leqslant d \quad (2.6)$$

for $x \in I \backslash \{x_\eta\}$. This can be accomplished as follows. Let $A = \{\xi_1, ..., \xi_l, x_\eta - \delta, x_\eta + \delta\}$, $\tau_1 = \min_{x \in A} g(x) > 0$, $\tau_2 = \max_{x \in A} g(x)$, $f_1(x) = \max\{\tau_1, R_0(x)^p - d/| B(x)|\}$, and $f_2(x) = \min\{\tau_2, R_0(x)^p + d/| B(x)|\}$. Then f_1 and f_2 are continuous on $I \backslash (x_\eta - \delta, x_\eta + \delta)$ and $f_1(x) \leqslant g(x) \leqslant f_2(x)$ for $x \in A$. By a variant of the Tietze extension theorem, g can be extended continuously to $I \backslash (x_\eta - \delta, x_\eta + \delta)$ so that $f_1(x) \leqslant g(x) \leqslant f_2(x)$ for $x \in I \backslash (x_\eta - \delta, x_\eta + \delta)$. Thus g is continuous on I and satisfies (2.5) for $x \in I$ and (2.6) for $x \in I \backslash \{x_\eta\}$.

We finally show that $f = gB$ does not have a best approximation from $V(p, n, m)$. By (2.4), (2.6), and the fact that $R_\epsilon \to R_0$ uniformly on $I \backslash (x_\eta - \delta, x_\eta + \delta)$,

$$\lim_{\epsilon \to 0^+} \| f - R_\epsilon{}^p B \| = d$$

and thus

$$\inf_{R \in R^+(n,m)} \| f - R^p B \| \leqslant d.$$

Now assume that f has a best approximation $(R^*)^p B$ from $V(p, n, m)$ where $R^* \in R^+(n, m)$. For $i = 1, ..., l$,

$$(-1)^i [f(\xi_i) - R^*(\xi_i)^p B(\xi_i)]$$
$$\leqslant \| f - (R^*)^p B \| \leqslant d = (-1)^i [f(\xi_i) - R_0(\xi_i)^p B(\xi_i)].$$

Thus $(-1)^i [R(\xi_i)^p - R_0(\xi_i)^p] B(\xi_i) \geqslant 0$, $i = 1, ..., l$. Hence, $\sigma(-1)^i [R^*(\xi_i)^p - R_0(\xi_i)^p] \geqslant 0$, $i = 1, ..., l$, where σ is the common sign of the $B(\xi_i)$. Therefore, $\sigma(-1)^i [R^*(\xi_i) - R_0(\xi_i)] \geqslant 0$, $i = 1, ..., l$. Letting $R^* = P^*/Q^*$ where $P^* \in \pi_n$, $Q^* \in \pi_m$, $P^* \geqslant 0$ and $Q^* > 0$ on I, we see that

$$\sigma(-1)^i [P^*(\xi_i)(\xi_i - x_\eta)^2 - Q^*(\xi_i)[K(\xi_i - x_\eta)^2 + 1]] \geqslant 0 \quad (2.7)$$

for $i = 1, \ldots, l$. Since $P^*(x)(x - x_n)^2 + Q^*(x)[K(x - x_n)^2 + 1] \in \Pi_{n+m}$, (2.7) implies that

$$P^*(x)(x - x_n)^2 - Q^*(x)[K(x - x_n)^2 + 1] \equiv 0.$$

Evaluation for $x = x_n$ yields $Q^*(x_n) = 0$ which is a contradiction. Thus f does not have a best approximation from $V(p, n, m)$.

We remark that if the zeros of B are only at a or b, then R_ϵ could have been chosen to be in $R^+(1, 1)$, $\epsilon > 0$. In this case, the result of Theorem 2 can be extended to $n \geqslant 1$ and $m \geqslant 1$.

3. Characterization and Uniqueness of Best Approximations

In this section, we state two characterization theorems and a uniqueness theorem for best approximations from $V(p, n, m)$. The development of these theorems is essentially the same as that on page 158–163 in Cheney [1] and, as a result, we omit the proofs.

Let $R \in R^+(n, m)$ and suppose that $g \notin R^+(n, m)$. Let

$$y_1 = \{x \in I: |f(x) - R(x)^p B(x)| = \|f - R^p B\|\},$$

and $y_2 = \{x \in I: R(x) = 0\}$. For $x \in I$, let $\sigma(x) = \operatorname{sgn}[g(x) - R(x)^p]$. Note that if $x \in y_2$, then $\sigma(x) = 1$. Let

$$U = \{\bar{P} - R\bar{Q}: P \in \pi_n \text{ and } Q \in \pi_m\}.$$

The first characterization theorem is of the Kolmogorov type and holds even if the approximants involve generalized rational functions as defined on p. 158 of [1] rather than rational functions.

Theorem 3. *Suppose $g \notin R^+(n, m)$. Then $R^p B$ is a best approximation to $f = gB$ from $V(p, n, m)$ if and only if there is no $\varphi \in U$ such that $\sigma(x)\, \varphi(x) > 0$ for all $x \in y_1 \cup y_2$.*

The second characterization theorem is of the alternation type. In light of Williams' characterization theorem [5] and the usual characterization results for restricted range approximation, this result is quite natural.

Theorem 4. *Suppose $R = P/Q \in R^+(n, m)$ where P/Q is a completely reduced representation for R and let*

$$d = 1 + \max\{n + \deg Q, m + \deg P\}$$

if $R \not\equiv 0$ and $d = n + 1$ if $R \equiv 0$. Then R^pB is a best approximation to f from $V(p, n, m)$ if and only if there exist $d + 1$ points $\xi_0 < \xi_1 < \cdots < \xi_d$ in I such that

(i) $|f(\xi_i) - R(\xi_i)^p B(\xi_i)| = \|f - R^pB\|$ *or* $R(\xi_i) = 0$, $i = 0,\ldots, d$, *and*

(ii) $\operatorname{sgn}[g(\xi_i) - R(\xi_i)^p] = -\operatorname{sgn}[g(\xi_{i-1}) - R(\xi_{i-1})^p]$, $i = 1,\ldots, d$.

The uniqueness of best approximations now follows directly from Theorem 4.

THEOREM 5. *The function $f = gB$ has at most one best approximation from $V(p, n, m)$.*

4. CONCLUSION

The principle results of this paper are that the existence theorem of Taylor and Williams [4] extends to the case $n \geqslant 0$ and that the conditions of this theorem are minimal when $n \geqslant 2$ and $m \geqslant 2$. In addition, the results of Section 3 indicate that Williams' characterization and uniqueness results [5] also extend to the more general setting of this paper. It would be of interest to investigate algorithms to find best approximations to $f = gB$ from $V(p, n, m)$.

ACKNOWLEDGMENT

The author would like to thank the referee for pointing out a simplification of the proof of Theorem 2.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
2. C. B. DUNHAM, Chebyshev approximation with respect to a vanishing weight function, *J. Approximation Theory* **12** (1974), 305–306.
3. C. B. DUNHAM, Rational approximation with a vanishing weight function and with a fixed value at zero, *Math. Comp.* **30** (1976), 45–47.
4. G. D. TAYLOR AND J. WILLIAMS, Existence questions for the problem of Chebyshev approximation by interpolating rationals, *Math. Comp.* **28** (1974), 1097–1103.
5. J. WILLIAMS, Numerical Chebyshev approximation by interpolating rationals, *Math. Comp.* **26** (1972), 199–206.